Equations

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Every equation of the lecture series in one place, ordered from fundamental physics to the highest abstraction. No derivations and almost no words — those live in the lectures — just the results, as a reference.

Fundamental physics

The QED Lagrangian — everything in electronics follows from this

$$\mathcal{L} = \bar{\psi}[i\hbar c\gamma^\mu \partial_\mu - mc^2]\psi - q[\bar{\psi}\gamma^\mu\psi]A_\mu - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$$

Schrödinger equation

$$i\hbar\frac{d}{dt}\psi(r,t) = \widehat{H}\psi(r,t)$$

Probability density of a particle

$$P = |\psi(r,t)|^2 \text{ , } \psi(r,t) = Ae^{i(kr-\omega t)}$$

Heisenberg uncertainty

$$\sigma_x\sigma_p \geq \frac{\hbar}{2} \text{ , } \Delta E\Delta t > \frac{h}{2\pi}$$

Fermi-Dirac distribution, and its Boltzmann tail

$$f(E) = \frac{1}{e^{(E-E_F)/kT}+1} \approx e^{(E_F-E)/kT}$$

Maxwell's equations

$$\oint_{\partial\Omega} \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\epsilon_0}\iiint_V \rho \cdot dV \text{ , } \oint_{\partial\Omega} \mathbf{B}\cdot d\mathbf{S} = 0$$

$$\oint_{\partial\Sigma} \mathbf{E}\cdot d\ell = -\frac{d}{dt}\iint_\Sigma \mathbf{B}\cdot d\mathbf{S}$$

$$\oint_{\partial\Sigma} \mathbf{B}\cdot d\ell = \mu_0\left(\iint_\Sigma \mathbf{J}\cdot d\mathbf{S} + \epsilon_0\frac{d}{dt}\iint_\Sigma \mathbf{E}\cdot d\mathbf{S}\right)$$

Force on a charge

$$\vec{F} = q\vec{E}$$

Semiconductors

Density of electrons in the conduction band

$$n = \int_{E_C}^{\infty} N(E)f(E)dE$$

Effective density of states

$$N_c = 2\left[\frac{2\pi kTm_n^*}{h^2}\right]^{3/2} \text{ , } N_v = 2\left[\frac{2\pi kTm_p^*}{h^2}\right]^{3/2}$$

Intrinsic carrier concentration

$$n_i = \sqrt{N_cN_v}\, e^{-E_g/(2kT)}$$

Doped silicon (mass action)

$$n_n = N_D \text{ , } p_n = \frac{n_i^2}{N_D} \text{ ; } p_p = N_A \text{ , } n_p = \frac{n_i^2}{N_A}$$

Drift current

$$\vec{J} = qn\mu\vec{E}$$

Diffusion current

$$J = -qD_n\frac{d\rho}{dx}$$

Thermal voltage

$$V_T = \frac{kT}{q} \approx 25.85 \text{ mV @ 300 K}$$

Diode

Built-in voltage of a pn junction

$$\Phi_0 = V_T \ln\left(\frac{N_AN_D}{n_i^2}\right)$$

Depletion width

$$W = \sqrt{\frac{2\varepsilon_{si}(\Phi_0+V_R)}{q}\cdot\frac{N_A+N_D}{N_AN_D}}$$

Diode equation

$$I_D = I_S\left(e^{V_D/V_T} - 1\right) \text{ , } I_S = qAn_i^2\left(\frac{D_n}{L_nN_A} + \frac{D_p}{L_pN_D}\right)$$

Forward voltage temperature dependence

$$V_D = \frac{kT}{q}(\ell - 3\ln T) + V_G \Rightarrow \frac{dV_D}{dT} = \frac{k}{q}(\ell - 3\ln T - 3)$$

Difference of two diode voltages (PTAT)

$$V_{D1} - V_{D2} = V_T \ln N$$

Generation (leakage) current of a reverse biased junction

$$I_{gen} = \frac{qAn_iW}{\tau_g}$$

MOSFET

Weak inversion

$$I_D = I_{D0}\frac{W}{L}e^{V_{eff}/nV_T} \text{ if } V_{DS} > 3V_T$$

$$n = \frac{C_{ox} + C_{j0}}{C_{ox}} \text{ , } I_{D0} = (n-1)\mu_n C_{ox} V_T^2$$

Strong inversion, define

$$V_{eff} = V_{GS} - V_{tn} \text{ , } \ell = \mu_n C_{ox}\frac{W}{L}$$

$$I_{DS} = \ell \begin{cases} V_{eff}V_{DS} & \text{if } V_{DS} << V_{eff} \\ V_{eff}V_{DS} - V_{DS}^2/2 & \text{if } V_{DS} < V_{eff} \\ \frac{1}{2}V_{eff}^2\left[1 + \lambda(V_{DS} - V_{eff})\right] & \text{if } V_{DS} > V_{eff} \end{cases}$$

Transconductance

$$g_m = \frac{\partial I_{DS}}{\partial V_{GS}} = \ell V_{eff} = \sqrt{2\ell I_D} = \frac{2I_D}{V_{eff}} \text{ (strong) , } g_m = \frac{I_D}{nV_T} \text{ (weak)}$$

Transconductance efficiency

$$\frac{g_m}{I_D} = \frac{1}{nV_T} \text{ (weak) , } \frac{g_m}{I_D} = \frac{2}{V_{eff}} \text{ (strong)}$$

Output conductance and intrinsic gain

$$g_{ds} = \frac{1}{r_{ds}} \approx \lambda I_D \text{ , } A = g_m r_{ds} = \frac{2}{\lambda V_{eff}}$$

Capacitances

$$C_{gs} = \frac{2}{3}WLC_{ox} \text{ (saturation) , } C_{gd} = C_{ox}WL_{ov}$$

$$C_{sb} = (A_s + A_{ch})C_{js} \text{ , } C_{js} = \frac{C_{j0}}{\sqrt{1 + \frac{V_{SB}}{\Phi_0}}}$$

Miller's theorem

$$C_{in} = (1 + A)C \text{ , } C_{out} = \left(1 + \frac{1}{A}\right)C \Rightarrow C_{in} \approx C_{gd}\, g_m r_{ds}$$

Matching (Pelgrom)

$$\sigma^2(\Delta P) = \frac{A_P^2}{WL} + S_P^2 D^2$$

Current mismatch (Kinget)

$$\frac{\sigma_{I_D}^2}{I_D^2} = \frac{1}{WL}\left[\left(\frac{g_m}{I_D}\right)^2\sigma_{vt}^2 + \frac{\sigma_\ell^2}{\ell^2}\right] \text{ , } \sigma_{v_i}^2 = \frac{\sigma_{I_D}^2}{g_m^2}$$

Noise

Mean square and power spectral density

$$\overline{x^2(t)} = \int_0^\infty S_x(f)df \text{ , } S_y(f) = S_x(f)|H(f)|^2$$

Thermal noise of a resistor

$$S_{th}(f) = 4kTR$$

Sampled (kT/C) noise bandwidth of an RC

$$f_x = \frac{\pi f_0}{2} = \frac{1}{4RC}$$

Uncorrelated sources add in power

$$\overline{e_{tot}^2} = \overline{e_1^2} + \overline{e_2^2}$$

Signal-to-noise ratio

$$SNR = 10\log\left(\frac{\overline{v_{sig}^2}}{\overline{e_n^2}}\right)$$

Noise factor and Friis' formula

$$F = \frac{SNR_{input}}{SNR_{output}} \text{ , } NF = 10\log(F)$$

$$F = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1G_2} + \cdots$$

CIRCUITS

Common source

$$A = -g_m r_{ds} \ , \ r_{out} = r_{ds}$$

Common drain (source follower)

$$A = \frac{g_m}{g_m + g_{ds} + g_s} \ , \ r_{out} \approx \frac{1}{g_m}$$

Common gate

$$A = 1 + g_m r_{ds} \ , \ r_{in} \approx \frac{1}{g_m}\left(1 + \frac{R_L}{r_{ds}}\right)$$

Source degeneration and cascode output resistance

$$r_{out} = r_{ds2}\left[1 + R_s(g_{m2} + g_{ds2})\right] \approx r_{ds2}\, g_{m2} R_s$$

REFERENCES AND BIAS

Bipolar / diode core

$$V_{BE} = V_T \ln\frac{I_C}{I_S} \text{ (CTAT)} \ , \ \Delta V_{BE} = V_T \ln N \text{ (PTAT)}$$

Brokaw bandgap output

$$V_{REF} = V_{BE3} + \frac{R_2}{R_3} V_T \ln\frac{R_2}{R_1}$$

Bandgap voltage with curvature

$$V_{BG} = V_{G0} + (m-1)\frac{kT}{q}\ln\frac{T_0}{T} + T\left[\frac{k}{q}\ln\frac{J_2}{J_1}\frac{2R_2}{R_1} - \frac{V_{G0} - V_{be0}}{T_0}\right]$$

FILTERS

Pole/zero frequency

$$\omega_{p|z} \propto \frac{1}{RC} \text{ (Active-RC)} \ , \ \omega_{p|z} \propto \frac{G_m}{C} \text{ (Gm-C)}$$

General biquad

$$H(s) = \frac{\frac{C_1}{C_B}s^2 + \frac{G_2}{C_B}s + \frac{G_1 G_3}{C_A C_B}}{s^2 + \frac{G_5}{C_B}s + \frac{G_3 G_4}{C_A C_B}}$$

SWITCHED CAPACITOR

SC resistance

$$Z_I = \frac{1}{C_1 f_\phi}$$

SC gain stage and integrator

$$H(z) = \frac{C_1}{C_2}z^{-1} \ , \ H(z) = \frac{C_1}{C_2}\frac{z^{-1}}{1 - z^{-1}}$$

First and second order IIR

$$H(z) = \frac{b}{z - a} \ , \ H(z) = \frac{bz}{z^2 - 2az + (a^2 + b^2)}$$

DATA CONVERTERS

Quantization noise

$$\overline{e_n^2} = \frac{\Delta^2}{12} \Rightarrow SQNR \approx 6.02B + 1.76 \text{ dB}$$

Oversampling

$$SQNR \approx 6.02B + 1.76 + 10\log(OSR)$$

First order noise shaping

$$Y(z) = STF(z)U(z) + NTF(z)E(z) \ , \ STF = z^{-1} \ , \ NTF = 1 - z^{-1}$$

$$SQNR = 6.02B + 1.76 - 5.17 + 30\log(OSR)$$

Figures of merit

$$FOM_W = \frac{P}{2^B f_s} \ , \ FOM_S = SNDR + 10\log\left(\frac{f_s/2}{P}\right)$$

DAC: a digital number scales a reference

$$V_{out} = D_{in} \times V_{ref}$$

VOLTAGE REGULATION

The inductor and capacitor of a switcher

$$I_x(t) = \frac{1}{L}\int V_x(t)dt \ , \ V_o(t) = \frac{1}{C}\int (I_x(t) - I_o(t))dt$$

Ideal buck output

$$V_o = V_{in} \times \text{Duty-Cycle}$$

PLL

A modulated carrier, and phase versus frequency

$$A_m(t)\cos\left(2\pi f_c t + \phi_m(t)\right) \ , \phi(t) = 2\pi \int_0^t f(t)dt$$

Loop gain of a charge-pump PLL

$$L(s) = \frac{K_{osc}K_{pd}K_{lp}H_{lp}(s)}{Ns}$$

$$K_{osc} = 2\pi\frac{df}{dV_{cntl}} \ , K_{pd} = \frac{I_{cp}}{2\pi} \ , K_{lp}H_{lp}(s) = \frac{1}{s(C_1+C_2)}\frac{1+sRC_1}{1+sR\frac{C_1C_2}{C_1+C_2}}$$

Oscillators

Crystal input impedance

$$Z_{in} \approx \frac{LC_F s^2 + 1}{LC_F C_P s^2 + C_F + C_P}$$

Ring oscillator frequency

$$f = \frac{1}{2N t_{pd}} \ , t_{pd} \approx RC \Rightarrow f = \frac{\mu_n(VDD - V_{th})}{\frac{4}{3}NL^2}$$

Current starved ring

$$f \approx \frac{I_{control}}{C\frac{VDD}{2}N}$$

Radio

Friis transmission (free space)

$$P_{RX} = \frac{P_{TX}}{D^2}\left[\frac{\lambda}{4\pi}\right]^2$$

Receiver sensitivity

$$P_{RX_{sens}} = -174 \text{ dBm} + 10\log_{10}(DR) + NF + E_b/N_0$$

Would you like to know more?

Every equation here is derived in its own chapter of this book; the section titles above point at them

Johns and Martin carry the same set with more algebra [1]

[1] D. Johns and K. Martin, *Analog integrated circuit design*. John Wiley & Sons, Inc., 1997.

Carsten Wulff received the M.Sc. and Ph.D. degrees in electrical engineering from the Department of Electronics and Telecommunication, Norwegian University of Science and Technology (NTNU), in 2002 and 2008, respectively. During his Ph.D. work at NTNU, he worked on open-loop sigma-delta modulators and analog-to-digital converters in nanoscale CMOS technologies. In 2006-2007, he was a Visiting Researcher with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON, Canada. Since 2008 he's been with Nordic Semiconductor in various roles from analog designer, to Wireless Group Manager, to currently Principle IC Scientist. From 2014-2017 he did a part time Post.Doc focusing on compiled, ultra low power, SAR ADCs in nanoscale technologies. He's also an Adjunct Associate Professor at NTNU. His present research interests includes analog and mixed-signal CMOS design, design of high-efficiency analog-to-digital converters and low-power wireless transceivers. He is the developer of Custom IC Compiler, a general purpose integrated circuit compiler, and makes the occational video on analog integrated circuits at https://www.youtube.com/@analogicus. For full CV see https://analogicus.com/markdown-cv/.