

# Noise

Carsten Wulff, carsten@wulff.no

## I. NOISE

Noise is a phenomena that occurs in all electronic circuits. It places a lower limit on the smallest signal we can use. Many now have super audio compact disc (SACD) players with 24bit converters, 24 bits is around  $2^{24} = 16.78$  Million different levels. If 5V is the maximum voltage, the minimum would have to be  $\frac{5V}{2^{24}} \approx 298nV$ . That level is roughly equivalent to the noise in a 50 Ohm resistor with a bandwidth of 96kHz. There exist an equation that relates number of bits to signal to noise ratio ?, the equation specifies that  $SNR = 6.02 * Bits + 1.76 = 146.24dB$ . As of 12.2005 the best digital to analog converter (DAC) that Analog Devices (a very big semiconductor company) has is a DAC with 120dB SNR, that equals around  $Bits = (120 - 1.76)/6.02 = 19.64$ . In other words, the last four bits of your SACD player is probably noise!

## II. STATISTICS

The mean of a signal  $x(t)$  is defined as

$$\overline{x(t)} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t) dt$$

The mean square of  $x(t)$  defined as

$$\overline{x^2(t)} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$

The variance of  $x(t)$  defined as

$$\sigma^2 = \overline{x^2(t)} - \overline{x(t)}^2$$

For a signals with a mean of zero the variance is equal to the mean square. The auto-correlation of  $x(t)$  is defined as

$$\begin{aligned} R_x(\tau) &= \overline{x(t)x(t+\tau)} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)x(t+\tau) dt \end{aligned}$$

## III. AVERAGE POWER

Average power is defined for a continuous system as ([eq:powcont]) and for discrete samples it can be defined as ([eq:powsamp]).

$P_{av}$  usually has the unit  $A^2$  or  $V^2$ , so we have to multiply/divide by the impedance to get the power in Watts. To get Volts and Amperes we use the root-mean-square (RMS) value which is defined as  $\sqrt{P_{av}}$ .

$$P_{av} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$

$$P_{av} = \frac{1}{N} \sum_{i=0}^N x^2(i)$$

If  $x(t)$  has a mean of zero then, according to ([eq:var]),  $P_{av}$  is equal to the variance of  $x(t)$ .

Many different notations are used to denote average power and RMS value of voltage or current, some of them are listed in Table [t:avgpow] and Table 2. Notation can be a confusing thing, it changes from book to book and makes expressions look different.

It is important to realize that it does not matter how you write average power and RMS value. If you want you can invent your own notation for average power and RMS value. However, if you are presenting your calculations to other people it is convenient if they understand what you have written. In the remainder of this paper we will use  $\overline{e_n^2}$  for average power when we talk about voltage noise source and  $\overline{i_n^2}$  for average power when we talk about current noise source. The n subscript is used to identify different sources and can be whatever.

Voltage	Current
$V_{rms}^2$	$I_{rms}^2$
$\overline{V_n^2}$	$\overline{I_n^2}$
$v_n^2$	$i_n^2$
Voltage	Current
$V_{rms}$	$I_{rms}$
$\sqrt{\overline{V_n^2}}$	$\sqrt{\overline{I_n^2}}$
$\sqrt{v_n^2}$	$\sqrt{i_n^2}$

## IV. NOISE SPECTRUM

With random noise it is useful to relate the average power to frequency. We call this Power Spectral Density (PSD). A PSD plots how much power a signal carries at each frequency. In literature  $S_x(f)$  is often used to denote the PSD. In the same way that we use  $V^2$  as unit of average power, the unit of the PSD is  $\frac{V^2}{Hz}$  for voltage and  $\frac{A^2}{Hz}$  current. The root spectral density is defined as  $\sqrt{S_x(f)}$  and has unit  $\frac{V}{\sqrt{Hz}}$  for voltage and  $\frac{A}{\sqrt{Hz}}$  for current.

The power spectral density is defined as two times the Fourier transform of the auto-correlation function ?

$$S_x(f) = 2 \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau$$

This can also be written as

$$\begin{aligned}
S_x(f) &= 2 \left[ \int_{-\infty}^{\infty} R_x(\tau) \cos(\omega\tau) d\tau - \int_{-\infty}^{\infty} R_x(\tau) j \sin(\omega\tau) d\tau \right] \\
&= 2 \left[ \int_{-\infty}^0 R_x(\tau) \cos(\omega\tau) d\tau + \int_0^{\infty} R_x(\tau) \cos(\omega\tau) d\tau \right] \\
&\quad - 2j \left[ \int_{-\infty}^0 R_x(\tau) \sin(\omega\tau) d\tau + \int_0^{\infty} R_x(\tau) \sin(\omega\tau) d\tau \right] \\
&= 4 \int_0^{\infty} R_x(\tau) \cos(\omega\tau) d\tau \\
&\quad - 2j \left[ - \int_0^{\infty} R_x(\tau) \sin(\omega\tau) d\tau + \int_0^{\infty} R_x(\tau) \sin(\omega\tau) d\tau \right] \\
&= 4 \int_0^{\infty} R_x(\tau) \cos(\omega\tau) d\tau
\end{aligned}$$

, since  $e^{-j\omega\tau} = \cos(\omega\tau) - j \sin(\omega\tau)$ ,  $R_x(\tau)$  and  $\cos(\omega\tau)$  are symmetric around  $\tau = 0$  while  $\sin(\omega\tau)$  is asymmetric around  $\tau = 0$ .

The inverse of power spectral density is defined as

$$R_x(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f\tau} df = \int_0^{\infty} S_x(f) \cos(\omega\tau) df$$

If we set  $\tau = 0$  we get

$$\overline{x^2(t)} = \int_0^{\infty} S_x(f) df$$

which means we can easily calculate the average power if we know the power spectral density. As we will see later it is common to express noise sources in PSD form.

Another very useful theorem when working with noise in the frequency domain is this

$$S_y(f) = S_x(f) |H(f)|^2$$

, where  $S_y(f)$  is the output power spectral density,  $S_x(f)$  is the input power spectral density and  $H(f)$  is the transfer function of a time-invariant linear system.

If we insert ([eq:psd\_hf]) into ([eq:ms\_psd]), with  $S_x(f) = a \text{ constant} = D_v$  we get

$$\overline{x^2(t)} = \int S_y(f) df = D_v \int |H(f)|^2 df = D_v f_x$$

, where  $f_x$  is what we call the noise bandwidth. For a single time constant RC network the noise bandwidth is equal to

$$f_x = \frac{\pi f_0}{2} = \frac{1}{4RC}$$

where  $f_x$  is the noise bandwidth and  $f_0$  is the 3dB frequency.

We haven't told you this yet, but thermal noise is white and white means that the power spectral density is flat (constant over all frequencies). If  $S_x(f)$  is our thermal noise source and  $H(f)$  is a standard low pass filter, then equation ([eq:psd\_hf])

tells us that the output spectral density will be shaped by  $H(f)$ . At frequencies above the  $f_x$  in  $H(f)$  we expect the root power spectral density to fall by 20dB per decade.

## V. PROBABILITY DISTRIBUTION

**Theorem 1** (Central limit theorem). *The sum of  $n$  independent random variables subjected to the same distribution will always approach a normal distribution curve as  $n$  increases.*

This is a neat theorem, it explains why many noise sources we encounter in the real world are white.<sup>1</sup> Take thermal noise for example, it is generated by random motion of carriers in materials. If we look at a single electron moving through the material the probability distribution might not be Gaussian. But summing probability distribution of the random movements with a large number of electrons will give us a Gaussian distribution, thus thermal noise is white.

## VI. PSD OF A WHITE NOISE SOURCE

If we have a true random process with Gaussian distribution we know that the autocorrelation function only has a value for  $\tau = 0$ . From equation ([eq:autocor]) we have that

$$\begin{aligned}
R_x(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)x(t-\tau) dt \\
&= \left[ \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt \right] \delta(\tau) \\
&= \overline{x^2(t)} \delta(\tau)
\end{aligned}$$

The reason being that in a true random process  $x(t)$  is uncorrelated with  $x(t+\tau)$  where  $\tau$  is an integer. If we use equation ([eq:psd]) we see that

$$\begin{aligned}
S_x(f) &= 2 \int_{-\infty}^{\infty} \overline{x^2(t)} \delta(\tau) e^{-j2\pi f\tau} d\tau \\
&= \frac{2\overline{x^2(t)}}{\int_{-\infty}^{\infty} \delta(\tau) e^{-j2\pi f\tau} d\tau} \\
&= \frac{2\overline{x^2(t)}}{1}
\end{aligned}$$

, since

$$\int \delta(\tau) e^{-j2\pi f\tau} d\tau = e^0 = 1$$

This means that the power spectral density of a white noise source is flat, or in other words, the same for all frequencies.

<sup>1</sup>Gaussian distribution = normal distribution. Noise sources with Gaussian distribution are called white

## VII. SUMMING NOISE SOURCES

Summing noise sources is usually trivial, but we need to know why and when it is not. We if we write the time dependant noise signals as

$$v_{tot}^2(t) = (v_1(t) + v_2(t))^2 = v_1^2(t) + 2v_1(t)v_2(t) + v_2^2(t)$$

The average power is defined as

$$\begin{aligned} \overline{e_{tot}^2} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} v_{tot}^2(t) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} v_1^2(t) dt \\ &+ \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} v_2^2(t) dt \\ &+ \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} 2v_1(t)v_2(t) dt \\ &= \overline{e_1^2} + \overline{e_2^2} + \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} 2v_1(t)v_2(t) dt \end{aligned}$$

If  $\overline{e_1^2}$  and  $\overline{e_2^2}$  are uncorrelated noise sources we can skip the last term in ([eq:noisesum]) and just write

$$\overline{e_{tot}^2} = \overline{e_1^2} + \overline{e_2^2}$$

Most natural noise sources are uncorrelated.

## VIII. SIGNAL TO NOISE RATIOS

Signal to Noise Ratio (SNR) is a common method to specify the relation between signal power and noise power in linear systems. It is defined as

$$\begin{aligned} SNR &= 10 \log \left( \frac{\text{Signal power}}{\text{Noise power}} \right) \\ &= 10 \log \left( \frac{\overline{v_{sig}^2}}{\overline{e_n^2}} \right) \\ &= 20 \log \left( \frac{v_{rms}}{\sqrt{\overline{e_n^2}}} \right) \end{aligned}$$

Another useful ratio is Signal to Noise and Distortion (SNDR), since most real systems exhibit non-linearities it is useful to include distortion in the ratio. One can calculate SNR and SNDR in many ways. If we don't know the expression for  $\overline{e_n^2}$  we can do a FFT of our output signal. From this FFT we sum spectral components except at the signal frequency to get noise and distortion. SNR is normally calculated as

$$SNR = 10 \log \left( \frac{\text{Signal power}}{\text{Noise power} - 6 \text{ first harmonics}} \right)$$

And SNDR is calculated as

$$SNDR = 10 \log \left( \frac{\text{Signal power}}{\text{Noise power}} \right)$$

## IX. NOISE FIGURE AND FRIIS FORMULA

Noise factor is a measure on the noise performance of a system. It is defined as

$$F = \frac{\overline{v_o^2}}{\text{source contribution to } \overline{v_o^2}}$$

where  $\overline{v_o^2}$  is the total output noise.

The noise figure is defined as (noise factor in dB)

$$NF = 10 \log(F)$$

The noise factor can also be defined as

$$F = \frac{SNR_{input}}{SNR_{output}}$$

This brings us right into what is known as Friis formula. If we have a multistage system, for example several amplifiers in cascade, the total noise figure of the system is defined as

$$F = 1 + F_1 - 1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} + \dots$$

Here  $F_i$  is the noise figures of the individual stages and  $G_i$  is the available gain of each stage. This can be rewritten as

$$F = F_1 + \sum_{i=1}^N \frac{F_{i+1} - 1}{\prod_{k=1}^{i-1} G_k}$$

Friis formula tells us that it is the noise in the first stage that is the most important if  $G_1$  is large. We could say that in a system it is important to amplify the noise as early as possible!

## X. SPECTRAL DENSITY

Warning: This is not an introduction to spectral density. If the subject is completely unfamiliar I'd advise reading another source. For example chapter 4 in ? or chapter 7 in ?.

### A. Definition of Spectral Density

There are two different definitions of spectral density used in the literature. They differ by a factor of two. The one used in signal processing books, like ?, is

$$S_{x1}(f) = \int_{-\infty}^{\infty} R_{x1}(\tau) e^{-j\omega\tau} d\tau$$

And the one often used in books about noise, like ?, is

$$S_{x2}(f) = 2 \int_{-\infty}^{\infty} R_{x2}(\tau) e^{-j\omega\tau} d\tau$$

In both cases  $R_{xi}(\tau)$  is the auto-correlation function defined as

$$R_{xi}(\tau) = \overline{x_i(t)x_i(t+\tau)}$$

As we can plainly see

$$S_{x1}(f) \neq S_{x2}(f)$$

, there is no way these two can be made equal if

$$R_{x1}(\tau) = R_{x2}(\tau)$$

This is ok, there is no problem having two different definitions for two different functions. In reality  $S_{x1}(f)$  and  $S_{x2}(f)$  are different functions of frequency, and we could say that

$$S_{x2}(f) = 2S_{x1}(f)$$

if ([eq:rxequal]) is true.

### B. Sources of Confusion

The problem with spectral density arises when reading literature from different communities, for example ? and ? where  $S_x(f)$  is used for both  $S_{x1}(f)$  and  $S_{x2}(f)$ . When I started investigating spectral densities this lead me to believe that different sources defined the same measure “spectral density” in two different ways. The more sources I investigated the more unsure I was about which of the two definitions that was correct. After months of searching (not actively, but sporadically) I eventually found the original source of the definition of spectral density ?. Having the original source helped, but I still don’t know when the original definition split into ([eq:psd1]) and ([eq:psd2]). However, I’m pretty sure it’s just a matter of convenience. To see why ([eq:psd2]) is the most common among sources concerning noise we look at the inverse Fourier Transform. By the way, if you had not noticed yet, ([eq:psd1]) says that *Spectral density is the Fourier Transform of the Auto-Correlation function*. The inverse Fourier Transform of ([eq:psd1]) is

$$R_{x1}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{x1}(f) e^{j\omega\tau} d\omega = \int_{-\infty}^{\infty} S_{x1}(f) e^{j\omega\tau} df$$

,since  $d\omega = df dw/df = 2\pi df$ . And for ([eq:psd2])

$$R_{x2}(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} S_{x2}(f) e^{j\omega\tau} df$$

Before we proceed let’s get rid of the  $e$ ’s. We know that  $e^{j\alpha} = \cos \alpha + j \sin \alpha$ . So we could rewrite ([eq:psd1]) as

$$S_{x1}(f) = \int_{-\infty}^{\infty} R_{x1}(\tau) [\cos(\omega\tau) + j \sin(\omega\tau)] d\tau$$

and it turns out that since  $R_{x1}(\tau)$  is an even function we can drop the  $j \sin \omega\tau$  term.  $S_{x1}(f)$  is also an even function since the Fourier Transform of an even function is even.

The definitions then become

$$S_{x1}(f) = \int_{-\infty}^{\infty} R_{x1}(\tau) \cos(\omega\tau) d\tau$$

$$R_{x1}(\tau) = \int_{-\infty}^{\infty} S_{x1}(f) \cos(\omega\tau) df$$

and

$$S_{x2}(f) = 2 \int_{-\infty}^{\infty} R_{x2}(\tau) \cos(\omega\tau) d\tau$$

$$R_{x2}(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} S_{x2}(f) \cos(\omega\tau) df$$

We can rewrite  $R_{x2}(\tau)$  as

$$R_{x2}(\tau) = \overline{x_2(t)x_2(t+\tau)} = \int_0^{\infty} S_{x2}(f) \cos(\omega\tau) df$$

and if  $\tau = 0$

$$\overline{x_2^2(t)} = \int_0^{\infty} S_{x2}(f) df$$

So using spectral density definition ([eq:psd2]) we see that average power (mean square value of  $x_2(t)$ ) is equal to the integral from 0 to infinity of the spectral density. If we use ([eq:psd1]) average power would be

$$\overline{x_1^2(t)} = 2 \int_0^{\infty} S_{x1}(f) df$$

But if  $R_{x1}(\tau) = R_{x2}(\tau)$  then

$$\overline{x_2^2(t)} = \overline{x_1^2(t)}$$

even though  $S_{x1}(f) \neq S_{x2}(f)$ .

Definition ([eq:psd1]) is called the two-sided spectral density and ([eq:psd2]) is called the one-sided spectral density.

### C. Example: Thermal Noise

The spectral density of thermal noise in electronic circuit should be known to anyone that has studied analog electronics. We normally define the voltage spectral density of thermal noise as

$$S_{th}(f) = 4kTR$$

where  $k$  is Boltzmann’s constant,  $T$  the temperature in Kelvin and  $R$  the resistance. But ([eq:othermal]) is the spectral density when it is defined as in ([eq:psd2]). If we were to use ([eq:psd1]) then the spectral density of thermal noise would be

$$S_{th}(f) = 2kTR$$

Both these spectral densities would give the same average power value if we use the inverse Fourier Transform of ([eq:psd1]) and ([eq:psd2]).<sup>2</sup>

#### D. Einstein: The source

In his 1914 paper ? Albert Einstein described, supposedly for the first time, the auto-correlation function and what we have come to know as the spectral density. He defined the auto-correlation function as

$$\mathfrak{M}(\Delta) = \overline{F(t)F(t + \Delta)}$$

and the intensity (spectral density) as

$$I(\theta) = \int_0^T \mathfrak{M}(\Delta) \cos(\pi \frac{\Delta}{\theta}) d\Delta$$

,where the period  $\theta = T/n$  and  $T$  is a very large value. The paper is very short, only 1 page, but it is worth reading. Note that ([eq:psd1]) is often referred to as the *Wiener-Khintchine* theorem.



**Carsten Wulff** received the M.Sc. and Ph.D. degrees in electrical engineering from the Department of Electronics and Telecommunication, Norwegian University of Science and Technology (NTNU), in 2002 and 2008, respectively. During his Ph.D. work at NTNU, he worked on open-loop sigma-

delta modulators and analog-to-digital converters in nanoscale CMOS technologies. In 2006-2007, he was a Visiting Researcher with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON, Canada. Since 2008 he's been with Nordic Semiconductor in various roles, from analog designer, to Wireless Group Manager, to currently Principle IC Scientist. He's also an Adjunct Associate Professor at NTNU. His present research interests includes analog and mixed-signal CMOS design, design of high-efficiency analog-to-digital converters and low-power wireless transceivers. He is the developer of Custom IC Compiler, a general purpose integrated circuit compiler.

<sup>2</sup>Note that if you calculate the average power of  $S_{th}(f)$  you'll get infinity. You have to include the bandwidth of the circuit you are considering for average power to have a finite value.